

# Conformastationary disk-haloes in Einstein-Maxwell gravity II. The physical interpretation of the halo

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The relativistic treatment of galaxies modelled as a rotating disk surrounded by a magnetised material halo is considered. The galactic halo is modelled by a magnetised mass-energy distribution described by the energy-momentum tensor of a general fluid in canonical form. All the dynamics quantities characterising the physical of the halo are expressed in exact form in terms of an arbitrary solution of the Laplace's equation. By way of illustration, a “generalisation” of the Kuzmin solution of the Laplace's equation is used. The motion of a charged particle on the halo region is described. All the relevant quantities and the motion of the charged particle show a reasonable physical behaviour.

## I. INTRODUCTION

In the observational context, many ambiguities still exist about the main constituents, geometry and dynamics (thermodynamics) of the disk-halos. However, there are several different observations which probe the galactic and surrounding galactic magnetic field. A current revision of the status of our knowledge about the magnetic fields in our Milky Way and in nearby star-forming galaxies is summarised in [1]. Additionally, a study of the disk and halo rotation are reported in [2], whereas that in [3] the possibility that magnetic fields can be generated in the outskirts of disks is studied. Solutions for the Einstein and Einstein-Maxwell Field Equations which are consistently applicable in the context of astrophysical remains a topical problem. Nevertheless, the effects of magnetic fields on the physical processes in galaxies and their disk-halo interaction have been scarcely considered in the past. The presence of the electric field on the dark matter halo models has been considered in [4] whereas the presence of electromagnetic field in the halo-disk system has been studied in [5, 6]. In the last mentioned works the gravitational sources are statics.

In a precedent paper [7] we considered the conventional treatment of galaxies modelled as a stationary (conformastationary) thin disk and, correspondingly, we associate the galactic halo with the region surrounding the disk. We found the physical quantities characterising the disk and we concluded that the disk is made of a well-behaved general relativistic magnetised source. In the present work we are continuing our research of the thin disk-halo systems by studying the physical content of the energy-momentum tensor of the halo obtained in such paper. As we used the inverse method (where a solution of the field equations is taken and then the energy-momentum tensor is obtained), we do not impose restriction on the physical properties of the material constituting the halo. Moreover, as it is well known that, all static spacetime can be obtained from one stationary, here we generalise the results for conformastatic disk-halos obtained in [5] for the special case when the electric potential vanishes.

Our results are compatibles with the presented in [4] on possible features of galactic halo. Moreover, the description of the motion of charged particles on disk here deduced is in agreement with the results of analysis of particles motion in the magnetised disks discussed in [8]. As far as we know, this is the first relativistic model describing analytically the halo of a rotating source in presence of a magnetic field. To study the dynamics quantities characterising the physics of the halo, in Section II, we express these in terms of an arbitrary solution of the Laplace's equation and then we calculate these for generalisation of the Kuzmin solution of the Laplace's equation. We describe the motion of a charged particle on the halo region in Section III. Finally we complete the paper with a discussion of the results in Section IV.

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## II. THE HALO OF THE GENERALISED KUZMIN-LIKE DISKS

In the precedent paper [7] we used the formalism presented in [5] to obtain an exact relativistic model describing a system composed of a thin disk surrounded by a magnetised halo in a conformastationary space-time background. However, although we solved the Einstein-Maxwell distributional field equations to study the complete system disk-halo, we only focused our attention on the physical analysis of the disk. In this section, we will keep unchanged the the explicit metric and magnetic potentials in terms of the generalised Kuzmin solution of the Laplace's equation presented in [7] and we calculate the principal physical quantities describing the halo. To do so, we again express the energy-momentum tensor of the halo in the canonical form

$$M_{\alpha\beta}^{\pm} = (\mu^{\pm} + P^{\pm})V_{\alpha}V_{\beta} + P^{\pm}g_{\alpha\beta} + Q_{\alpha}^{\pm}V_{\beta} + Q_{\beta}^{\pm}V_{\alpha} + \Pi_{\alpha\beta}^{\pm}. \quad (1)$$

Consequently, we can say that the halo is constituted by a some mass-energy distribution described by the last energy-momentum tensor and  $V^{\alpha}$  is the four-velocity of certain observer. Accordingly  $\mu^{\pm}$ ,  $P^{\pm}$ ,  $Q_{\alpha}^{\pm}$  and  $\Pi_{\alpha\beta}^{\pm}$  are then the energy density, the isotropic pressure, the heat flux and the anisotropic tensor on the halo, respectively. Thus, it is immediate to see that

$$\mu^{\pm} = M_{\alpha\beta}^{\pm}V^{\alpha}V^{\beta}, \quad (2a)$$

$$P^{\pm} = \frac{1}{3}\mathcal{H}^{\alpha\beta}M_{\alpha\beta}^{\pm}, \quad (2b)$$

$$Q_{\alpha}^{\pm} = -\mu^{\pm}V_{\alpha} - M_{\alpha\beta}^{\pm}V^{\beta}, \quad (2c)$$

$$\Pi_{\alpha\beta}^{\pm} = \mathcal{H}_{\alpha}^{\mu}\mathcal{H}_{\beta}^{\nu}(M_{\mu\nu}^{\pm} - P^{\pm}\mathcal{H}_{\mu\nu}), \quad (2d)$$

with  $\mathcal{H}_{\mu\nu} \equiv g_{\mu\nu} + V_{\mu}V_{\nu}$  and  $\alpha = (t, r, \varphi)$ . The observer comoving with the fluid described by the energy-momentum tensor (1) will use the tetrad  $\{V^{\alpha}, I^{\alpha}, K^{\alpha}, Y^{\alpha}\} \equiv \{h_{(t)}^{\alpha}, h_{(r)}^{\alpha}, h_{(z)}^{\alpha}, h_{(\varphi)}^{\alpha}\}$ , with the corresponding dual tetrad  $\{V_{\alpha}, I_{\alpha}, K_{\alpha}, Y_{\alpha}\} \equiv \{-h_{(t)\alpha}^{(t)}, h_{(r)\alpha}^{(r)}, h_{(z)\alpha}^{(z)}, h_{(\varphi)\alpha}^{(\varphi)}\}$ , where

$$h_{(t)\alpha}^{(t)} = F^{1/2}\{1, 0, 0, \omega\}, \quad (3a)$$

$$h_{(r)\alpha}^{(r)} = F^{-\beta/2}\{0, 1, 0, 0\}, \quad (3b)$$

$$h_{(z)\alpha}^{(z)} = F^{-\beta/2}\{0, 0, 1, 0\}, \quad (3c)$$

$$h_{(\varphi)\alpha}^{(\varphi)} = F^{-\beta/2}\{0, 0, 0, r\}. \quad (3d)$$

The dual vector  $h_{(\alpha)}^{\beta}$  is obtained by the condition  $\eta_{(\alpha)(\beta)} = g_{\mu\nu}h_{(\alpha)}^{\mu}h_{(\beta)}^{\nu}$ , where  $F \equiv e^{2\phi}$  and  $e^{(1+\beta)\phi} = 1/(1-U)$ , being  $U$  a solution of the Laplace's equation. By using (2) and (A2) we obtain for the energy density and the pressure of the halo

$$\mu^{\pm} = \frac{(U_{,r}^2 + U_{,z}^2)e^{2(1+2\beta)\phi}}{(1+\beta)^2r^2} \left\{ (2\beta + \beta^2)r^2 - \frac{(1+\beta)^2}{2k^2}r^2e^{-2\phi} + \frac{3k_{\omega}^2(1+\beta)^2}{4} \right\} \quad (4)$$

and

$$P^{\pm} = \frac{(U_{,r}^2 + U_{,z}^2)e^{2(1+2\beta)\phi}}{3(1+\beta)^2r^2} \left\{ (4 - 2\beta - \beta^2)r^2 - \frac{(1+\beta)^2}{2k^2}r^2e^{-2\phi} + \frac{k_{\omega}^2(1+\beta)^2}{4} (1 + 3k_{\omega}^2r^{-2}U^2e^{2\beta\phi}(1 - e^{2\phi})) \right\}, \quad (5)$$

respectively. In the same way, by inserting (A2) into (2) we obtain for the heat flux of the halo

$$Q_{\alpha}^{\pm} = \frac{k_{\omega}e^{(1+2\beta)\phi}}{2(1+\beta)r} \left\{ 2(1+\beta)U_{,r} - (3+\beta)r(U_{,r}^2 + U_{,z}^2)e^{(1+\beta)\phi} \right\} \delta_{\alpha}^{\varphi}, \quad (6)$$

moreover, it is easy to see that the anisotropic tensor read

$$\Pi_{\alpha\beta}^{\pm} = P_r^{\pm}I_{\alpha}I_{\beta} + P_z^{\pm}K_{\alpha}K_{\beta} + P_{\varphi}^{\pm}Y_{\alpha}Y_{\beta} + 2P_T^{\pm}I_{(\alpha}K_{\beta)} \quad (7)$$

where

$$P_r^{\pm} = e^{2\beta\phi}\Pi_{rr}^{\pm}, \quad (8)$$

$$P_z^{\pm} = e^{2\beta\phi}\Pi_{zz}^{\pm}, \quad (9)$$

$$P_{\varphi}^{\pm} = \frac{e^{2\beta\phi}}{r^2}\Pi_{\varphi\varphi}^{\pm}, \quad (10)$$

$$P_T^{\pm} = e^{2\beta\phi}\Pi_{rz}^{\pm}. \quad (11)$$

and

$$\begin{aligned} \Pi_{rr}^{\pm} = & \frac{e^{2(1+\beta)\phi}}{3(1+\beta)^2 r^2} \left\{ \left( \frac{k_{\omega}^2(1+\beta)^2}{2} + \frac{2(1+\beta)^2}{k^2} r^2 e^{-2\phi} - 4(1+\beta-\beta^2)r^2 - \frac{3k_{\omega}^4(1+\beta)^2}{4} r^{-2} U^2 e^{2\beta\phi}(1-e^{2\phi}) \right) U_r^2 \right. \\ & + \left( -k_{\omega}^2(1+\beta)^2 - \frac{(1+\beta)^2}{k^2} r^2 e^{-2\phi} + 2(1+\beta-\beta^2)r^2 - \frac{3k_{\omega}^4(1+\beta)^2}{4} r^{-2} U^2 e^{2\beta\phi}(1-e^{2\phi}) \right) U_z^2 \\ & \left. - 3(1-\beta^2)r^2 e^{-(1+\beta)\phi} U_{,rr} \right\}, \end{aligned} \quad (12a)$$

$$\begin{aligned} \Pi_{zz}^{\pm} = & \frac{e^{2(1+\beta)\phi}}{3(1+\beta)^2 r^2} \left\{ \left( -k_{\omega}^2(1+\beta)^2 - \frac{(1+\beta)^2}{k^2} r^2 e^{-2\phi} + 2(1+\beta-\beta^2)r^2 - \frac{3k_{\omega}^4(1+\beta)^2}{4} r^{-2} U^2 e^{2\beta\phi}(1-e^{2\phi}) \right) U_r^2 \right. \\ & + \left( \frac{k_{\omega}^2(1+\beta)^2}{2} + \frac{2(1+\beta)^2}{k^2} r^2 e^{-2\phi} - 4(1+\beta-\beta^2)r^2 - \frac{3k_{\omega}^4(1+\beta)^2}{4} r^{-2} U^2 e^{2\beta\phi}(1-e^{2\phi}) \right) U_z^2 \\ & \left. - 3(1-\beta^2)r^2 e^{-(1+\beta)\phi} U_{,zz} \right\}, \end{aligned} \quad (12b)$$

$$\begin{aligned} \Pi_{\varphi\varphi}^{\pm} = & \frac{(U_{,r}^2 + U_{,z}^2)e^{2(1+\beta)\phi}}{3(1+\beta)^2} \left\{ 2(1+\beta-\beta^2)r^2 - \frac{(1+\beta)^2}{k^2} r^2 e^{-2\phi} + \frac{k_{\omega}^2(1+\beta)^2}{2} (1 + 3k_{\omega}^2 r^{-2} U^2 e^{2\beta\phi}(1-e^{2\phi})) \right\} \\ & - \frac{(1-\beta)}{1+\beta} r e^{(1+\beta)\phi} U_{,r}, \end{aligned} \quad (12c)$$

$$\Pi_{rz}^{\pm} = \frac{e^{2(1+\beta)\phi}}{(1+\beta)^2} \left\{ \left( -2(1+\beta-\beta^2) + \frac{(1+\beta)^2}{k^2} e^{-2\phi} + \frac{k_{\omega}^2(1+\beta)^2}{2r^2} \right) U_{,r} U_{,z} - (1-\beta^2) e^{-(1+\beta)\phi} U_{,rz} \right\}. \quad (12d)$$

Notice that  $\mathcal{P}^{\pm} \equiv P_r^{\pm} + P_z^{\pm} + P_{\varphi}^{\pm} = 0$  and consequently the trace  $\Pi^{\pm\alpha}_{\alpha} = 0$ . We have obtained expressions for the energy, pressure and the another quantities characterising the dynamic of the halo. All the dynamic quantities have been expressed in terms of an arbitrary  $U(r, z)$  solutions of the Laplace's equation. Consequently, we have as many halo models as disk-like solutions of the Laplace's equation. With the aim of describe the halo surrounding the generalised Kuzmin-like disks presented in [7] we consider the solution of the Laplace's equation in the form [9],

$$U = - \sum_{n=0}^N \frac{b_n P_n(z/R)}{R^{n+1}}, \quad P_n(z/R) = (-1)^n \frac{R^{n+1}}{n!} \frac{\partial^n}{\partial z^n} \left( \frac{1}{R} \right), \quad (13)$$

$P_n = P_n(z/R)$  being the Legendre polynomials in cylindrical coordinates which has been derived in the present form by a direct comparison of the Legendre polynomial expansion of the generating function with a Taylor expansion of  $1/r$ , the radius  $R$  denoted as  $R^2 \equiv r^2 + z^2$  and  $b_n$  arbitrary constant coefficients. Thus, the corresponding magnetic potential is

$$A_{\varphi} = -\frac{1}{k} \sum_{n=0}^N b_n \frac{(-1)^n}{n!} \frac{\partial^n}{\partial z^n} \left( \frac{z}{R} \right) \quad (14)$$

where we have imposed  $A_{\varphi}(0, z) = 0$  in order to preserve the regularity on the axis of symmetry, and, to introduce the corresponding discontinuity in the first-order derivatives of the metric potential and the magnetic potential required to define the disk we perform the transformation  $z \rightarrow |z| + a$ . To illustrate the results, we consider the two first members ( $N = 0, 1$ ) of the family of the generalised Kuzmin-like disks (as shown in 13). In Fig. 1, we show the behaviour of energy densities  $\mu^{\pm}$  on the halo as a function of  $r$  and  $z$ . In each case, we plot  $\mu_0^{\pm}(r, z)$  (Fig. 1(a)) and  $\mu_1^{\pm}(r, z)$  (Fig. 1(b)) for the indicate values of the parameters. It can be seen that the surface energy density is everywhere positive and it vanishes sufficiently fast as  $r$  increases. In Fig. 2, we show the behaviour of pressure  $P^{\pm}$  on the halo as a function of  $r$  and  $z$ . In each case, we plot  $P_0^{\pm}(r, z)$  (Fig. 2(a)) and  $P_1^{\pm}(r, z)$  (Fig. 2(b)) for the indicate values of the parameters. We can see that pressure are always positive and behave as the energy density of the halo. We can see that the behaviour of these quantities described here are in agreement with the results published in [4]. We also computed these functions for other values of the parameters within the allowed range and in all cases we have found a similar behaviour.

### III. MOTION OF A CHARGED TEST PARTICLE IN THE HALO

It is interesting to describe the motion of a particle “falling” in the halo, this kind of motion is called electrogeodesic. Following [10], the equation of motion of a charged particle in a gravitational and electromagnetic fields (electrogeodesic

equation) is obtained by

$$\frac{dv^\alpha}{ds} + \Gamma_{\beta\gamma}^\alpha v^\beta v^\gamma = \frac{e}{m} g^{\alpha\mu} F_{\mu\lambda} v^\lambda, \quad (15)$$

where  $e$  and  $m$  are the charge and the mass of the particle, respectively. The velocity of the particle as measured by the local observers is given by (see Appendix B in [7])  $v^\alpha = v^t(t^\alpha + \Omega\varphi^\alpha)$ , where

$$v^t = \frac{(1 - U)^{1/(1+\beta)}}{(1 + k_\omega U \Omega) \sqrt{1 - v^2}}. \quad (16)$$

Here, the 3-velocity  $v$  and the angular velocity  $\Omega$  of the particle as measured by the local observers are given by

$$v = \frac{r\Omega(1 - U)}{1 + k_\omega U \Omega} \quad (17)$$

and

$$\begin{aligned} \Omega &= \frac{k_\omega(U_{,r}^2 + U_{,z}^2) \left( (1 + \beta) + \frac{2U}{1-U} \right) \pm \sqrt{(U_{,r}^2 + U_{,z}^2)D}}{2(1 + \beta)r(1 - U)^2 U_{,r} - 2(U_{,r}^2 + U_{,z}^2)A}, \\ D &= 4(1 + \beta)r(1 - U)U_{,r} + (U_{,r}^2 + U_{,z}^2)(k_\omega^2(1 + \beta)^2 - 4\beta r^2), \\ A &= \beta r^2(1 - U) + k_\omega^2 U \left( 1 + \beta + \frac{U}{1 - U} \right), \end{aligned} \quad (18)$$

respectively. All the quantities depend on  $r$  and  $z$ . In Fig. 3(a) and Fig. 4(a) we show the behaviour of the velocity  $v^2_0$  and  $v^2_1$  of a charged particle following an electrogeodesic motion on the halo for the values of indicated parameters, respectively. Additionally, in Fig. 3(b) and Fig. 4(b), we plot the  $z$ -slices of the surface plot of the velocity and  $v^2_0$  and  $v^2_1$  for the indicated values of the parameters, respectively. These curves are obtained via vertical slices of the surface  $v^2 = v^2(r, z)$  (a vertical slice is a curve formed by the intersection of the surface  $v^2 = v^2(r, z)$  with the vertical planes). For each curve, we can see that the velocity is always less than 1, its maximum occurs around  $r = 0$ , and it vanishes sufficiently fast as  $r$  increases. We also computed these functions for other values of the parameters within the allowed range and in all cases we found a similar behaviour. Naturally, the description of the motion of charged particles on disk here deduced is in agreement with the results of analysis of the electrogeodesic motion of the particle in the magnetised disks discussed in [8].

#### IV. CONCLUDING REMARKS

In the present work we continued our research on the relativistic description of the disk surrounded by a halo in presence of an electromagnetic field. As can be observed, we considered the conventional treatment of galaxies modelled by thin disk and, correspondingly, we associate the galactic halo with the region surrounding the disk. We research started with the general formalism for a conformastatic spacetime in [5] and then we generalized the formalism to the conformastationary case in [7]. Here, we presented a physical description of the energy-momentum tensor of the halo. We concluded that the disk is made of a well-behaved general relativistic source surrounded by a well-behaved magnetized halo “material”.

As we used the inverse method, no restriction was imposed on the physical properties of the material constituting the halo. In fact, we expressed the energy-momentum tensor of the halo in the canonical form. The results obtained here are all consistent with the assumptions in the precedent paper [7] and generalise our results presented in [5]. Accordingly, when the parameter  $\beta$  in the metric is equal to one the usual conformastationary line element is obtained and then the pressure and the anisotropic tensor on the material constituting the halo disappear. In a similar way, when the parameter  $k_\omega$  is equal to zero, the heat flux on the halo vanishes, a feature of the static systems. Furthermore, when we take simultaneously  $k_\omega = 0$  and  $\beta = 1$ , the results presented here describe the halo of the disks presented in [5] for the special case when the electric potential vanishes. The results presented here are compatible with the results in the relativistic models of perfect fluid disks in a magnetic field presented in [8] and with the description of galactic halo presented in [4].

To analyse the physical content of the energy-momentum tensor of halo we expressed in the canonical form and we projected it in a comoving frame defined through the local observers tetrad. Accordingly, we found the explicit expressions for the energy density, pressure, heat flux, anisotropic tensor and electromagnetic potential of the halo

in terms a solution of the Laplaces equation. Although we used here a generalisation of the Kuzmin solution of the Laplaces equation, another models of disk-halos can be generated from the solutions presented here through suitable elections of the infinite family of solutions the Laplaces equation. Our result presented here has nothing to say about dark matter and exotic matter around of the disk. We think that its possible to use the models presented here as start point to generate realistic models of relativistic galaxies.

### Appendix A: The energy-momentum tensor of the halo

Following the results presented in [7], for the metric

$$ds^2 = -e^{2\phi}(dt + \omega d\varphi)^2 + e^{-2\beta\phi}(dr^2 + dz^2 + r^2 d\varphi^2), \quad e^{(1+\beta)\phi} = \frac{1}{1-U}, \quad \nabla^2 U = 0, \quad (\text{A1})$$

the non-zero components of the energy-momentum tensor of the halo are

$$M_{tt}^{\pm} = \frac{U_{,r}^2 + U_{,z}^2}{(1+\beta)^2} e^{4(1+\beta)\phi} \left\{ (\beta^2 + 2\beta) - \frac{1}{2k^2} (1+\beta)^2 e^{-2\phi} + \frac{3k_\omega^2}{4} (1+\beta)^2 r^{-2} \right\} \quad (\text{A2a})$$

$$M_{t\varphi}^{\pm} = \frac{k_\omega(U_{,r}^2 + U_{,z}^2)}{(1+\beta)^2} e^{4(1+\beta)\phi} \left\{ \frac{1}{2} (3+\beta)(1+\beta) e^{-(1+\beta)\phi} + U \left( \frac{3k_\omega^2}{4} (1+\beta)^2 r^{-2} + \beta^2 + 2\beta \right) \right\} \\ - k_\omega e^{2(1+\beta)\phi} \left( \frac{1}{2k^2} e^{2\beta\phi} U (U_{,r}^2 + U_{,z}^2) + r^{-1} U_{,r} \right) \quad (\text{A2b})$$

$$M_{rr}^{\pm} = -\frac{U_{,r}^2 - U_{,z}^2}{(1+\beta)^2} e^{2(1+\beta)\phi} \left( (2 - \beta^2) - \frac{(1+\beta)^2}{2k^2} e^{-2\phi} \right) + \frac{(1-\beta)}{(1+\beta)^2} e^{2(1+\beta)\phi} \left( 2U_{,r}^2 - (1+\beta) e^{-(1+\beta)\phi} U_{,rr} \right) \\ + \frac{k_\omega^2}{4} r^{-2} e^{2(1+\beta)\phi} (U_{,r}^2 - U_{,z}^2) \quad (\text{A2c})$$

$$M_{rz}^{\pm} = -\frac{2U_{,r}U_{,z}}{(1+\beta)^2} e^{2(1+\beta)\phi} \left( (2 - \beta^2) - \frac{(1+\beta)^2}{2k^2} e^{-2\phi} \right) + \frac{(1-\beta)}{(1+\beta)^2} e^{2(1+\beta)\phi} \left( 2U_{,r}U_{,z} - (1+\beta) e^{-(1+\beta)\phi} U_{,rz} \right) \\ + \frac{1}{2} k_\omega^2 r^{-2} e^{2(1+\beta)\phi} U_{,r}U_{,z}, \quad (\text{A2d})$$

$$M_{zz}^{\pm} = \frac{U_{,r}^2 - U_{,z}^2}{(1+\beta)^2} e^{2(1+\beta)\phi} \left( (2 - \beta^2) - \frac{(1+\beta)^2}{2k^2} e^{-2\phi} \right) + \frac{(1-\beta)}{(1+\beta)^2} e^{2(1+\beta)\phi} \left( 2U_{,z}^2 - (1+\beta) e^{-(1+\beta)\phi} U_{,zz} \right) \\ - \frac{1}{4} k_\omega^2 r^{-2} e^{2(1+\beta)\phi} (U_{,r}^2 - U_{,z}^2), \quad (\text{A2e})$$

$$M_{\varphi\varphi}^{\pm} = \frac{r^2(U_{,r}^2 + U_{,z}^2)}{(1+\beta)^2} e^{2(1+\beta)\phi} \left( (2 - \beta^2) - \frac{(1+\beta)^2}{2k^2} e^{-2\phi} \right) - \frac{(1-\beta)rU_{,r}}{(1+\beta)} e^{(1+\beta)\phi} + k_\omega^2 e^{2(1+\beta)\phi} \mathcal{K}, \quad (\text{A2f}) \\ \mathcal{K} = \frac{U_{,r}^2 + U_{,z}^2}{(1+\beta)^2} \left\{ (\beta^2 + 2\beta)U e^{2(1+\beta)\phi} + (3+\beta)(1+\beta)U e^{(1+\beta)\phi} + \frac{(1+\beta)^2 U^2}{2} e^{2\beta\phi} \left( \frac{3}{2} k_\omega^2 r^{-2} - \frac{1}{k^2} \right) + \frac{(1+\beta)^2}{4} \right\} \\ - 2r^{-1} U U_{,r},$$

where all the quantities depend on  $r$  and  $z$ .

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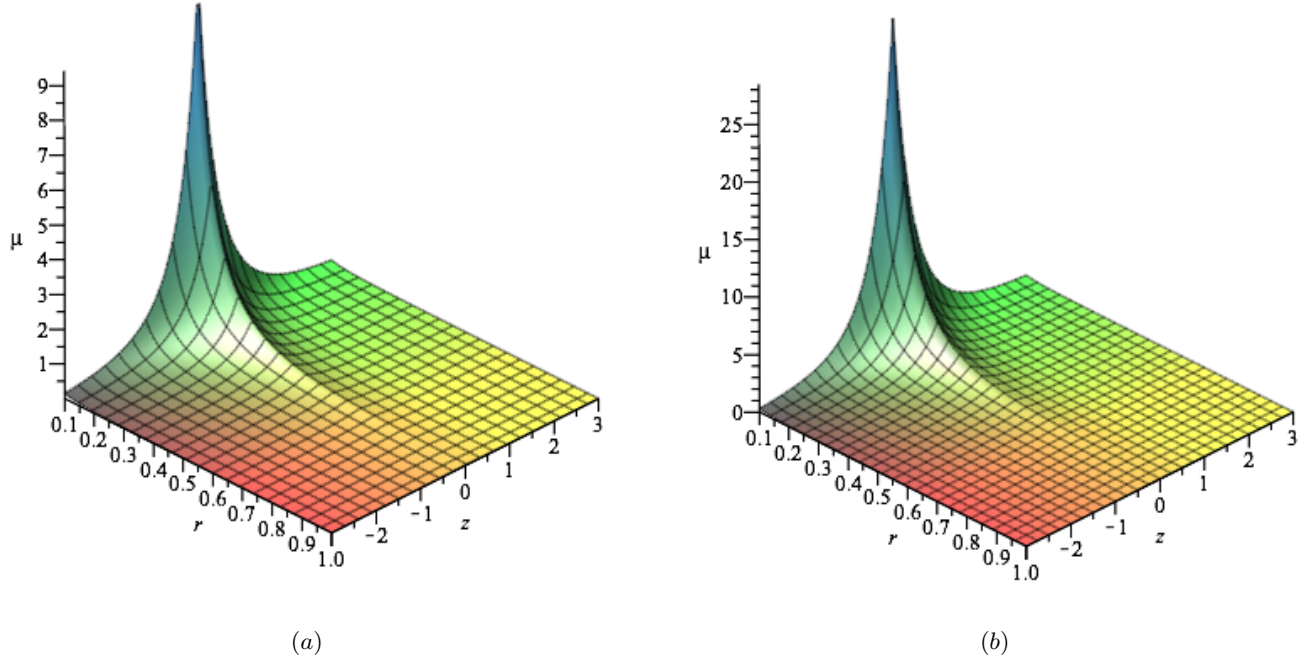


FIG. 1. Surface plots of the energy density (a)  $\mu_0^\pm$  and (b)  $\mu_1^\pm$  on the exterior halo as a functions depending on  $r$  and  $z$  with parameters  $a = b_0 = b_1 = k = k_\omega = 1$  and  $\beta = 0.75$

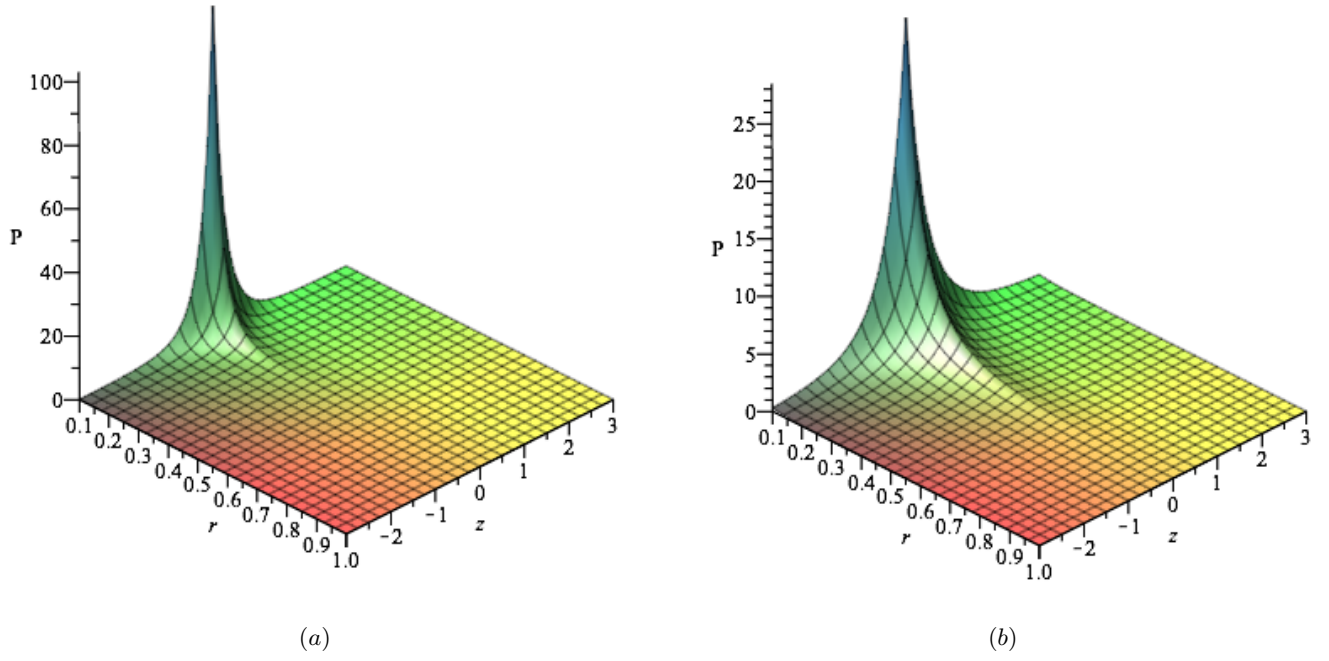


FIG. 2. Surface plots of the pressure (a)  $P_0^\pm$  and (b)  $P_1^\pm$  on the exterior halo as a functions depending on  $r$  and  $z$  with parameters  $a = b_0 = b_1 = k = k_\omega = 1$  and  $\beta = 0.75$

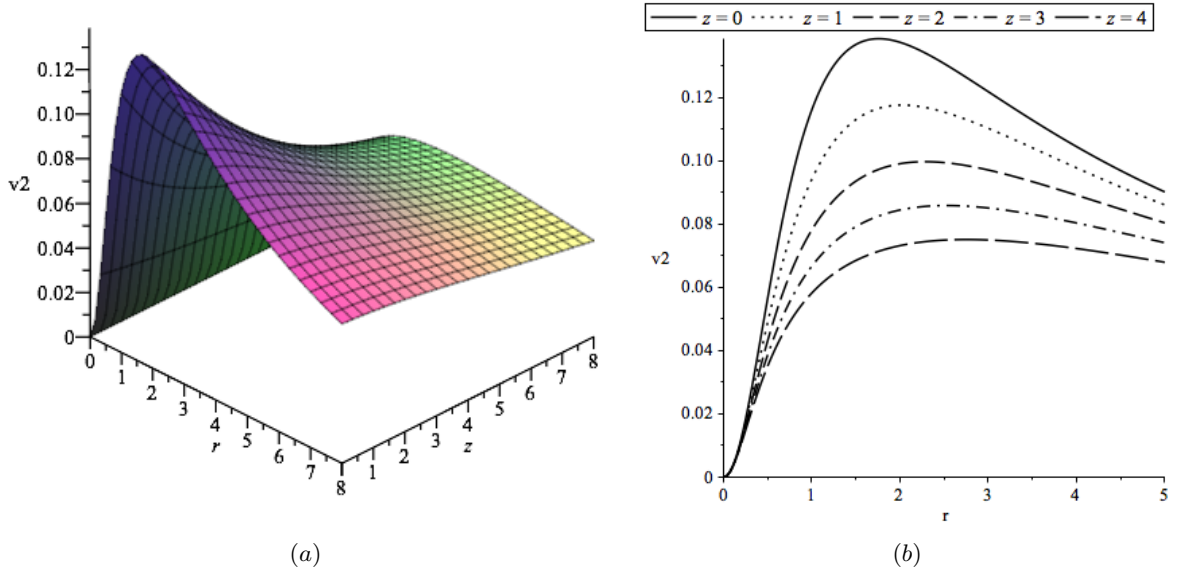


FIG. 3. Surface plot of the velocity (a)  $v_0^2$  and  $z$ -slices of the surface plot of the velocity (b) on the exterior halo as a functions depending on  $r$  and  $z$  with parameters  $a = b_0 = b_1 = k = k_\omega = 1$  and  $\beta = 0.75$

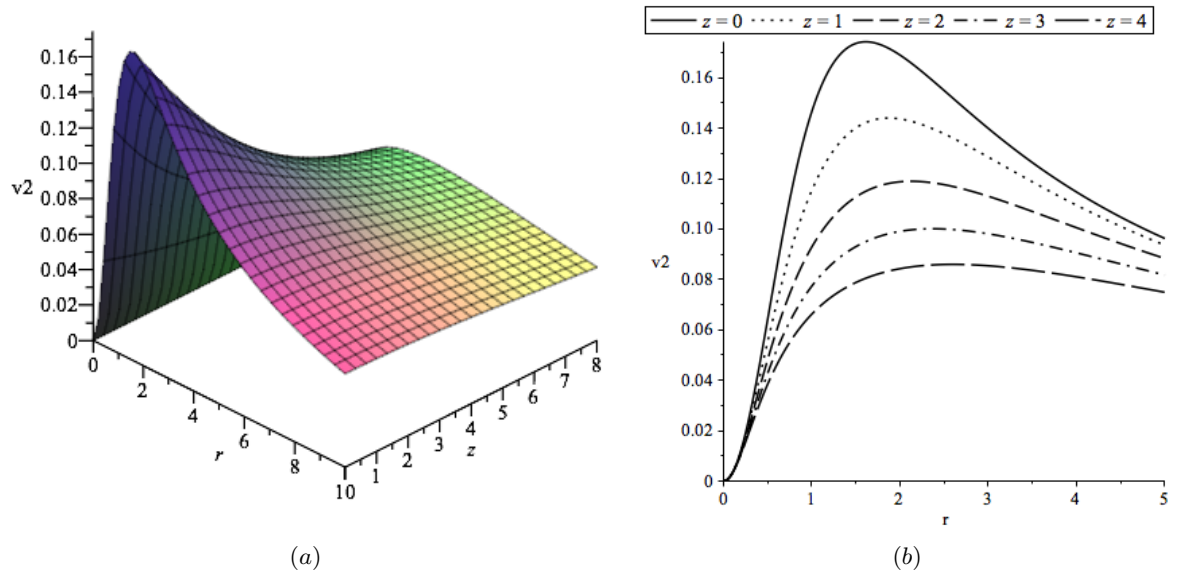


FIG. 4. Surface plot of the velocity (a)  $v_1^2$  and  $z$ -slices of the surface plot of the velocity (b) on the exterior halo as a functions depending on  $r$  and  $z$  with parameters  $a = b_0 = b_1 = k = k_\omega = 1$  and  $\beta = 0.75$